

GAPS FOR GEOMETRIC GENERA

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ABSTRACT. We investigate the possible values for geometric genera of subvarieties in a smooth projective variety. Values which are not attained are called *gaps*. For curves on a very general surface in \mathbb{P}^3 , the initial gap interval was found by Xu (see [7]), and the next one in our previous paper [4], where also the finiteness of the set of gaps was established and an asymptotic upper bound of this set was found. In the present paper we extend some of these results to smooth projective varieties of arbitrary dimension using a different approach.

INTRODUCTION

We consider the following problem. Let X be a smooth complex projective variety of dimension $n > 1$, with an ample divisor L . For each positive integer $s < n$, describe the set $\mathcal{P}_{X,s}$ of geometric genera of irreducible subvarieties $V \subset X$ of dimension s , and, in particular, the subset $\mathcal{P}_{X,L,s} \subseteq \mathcal{P}_{X,s}$ of geometric genera of irreducible complete intersections of $n - s$ hypersurfaces from $\bigcup_{m \geq 1} |mL|$. The complement of each of these sets in $\{0\} \cup \mathbb{N}$ is the corresponding *set of s -gaps*, and its maximal intervals are called *s -gap intervals*. For curves on a very general surface X in \mathbb{P}^3 of degree d (i.e., $n = 2$, $s = 1$) with a natural polarization $\mathcal{O}_X(1)$ the two sets $\mathcal{P}_{X,\mathcal{O}_X(1),1}$ and $\mathcal{P}_{X,1}$ coincide; the initial gap interval was found in [7] and the next one in [4]. In this case there exists a maximum G_d for the set of gaps ([4, Thm. 2.4] and Remark 3.2 below). This means that a very general surface of degree d in \mathbb{P}^3 carries a curve of geometric genus g for any $g > G_d$. In the present note we show that the latter remains true for any smooth projective variety, and in particular, for *any* (not just for a very general) smooth surface of degree d in \mathbb{P}^3 . One of our main results is the following:

Theorem 0.1. *Let X be an irreducible, smooth, projective variety of dimension $n > 1$, let L be a very ample divisor on X and let $s \in \{1, \dots, n - 1\}$. Then there is an integer $p_{X,L,s}$ (depending on X , L and s) such that for any $p \geq p_{X,L,s}$ one can find an irreducible subvariety Y of X of dimension s with at most ordinary points of multiplicity $s + 1$ as singularities such that $p_g(Y) = p$. Moreover, one can choose Y to be a complete intersection $Y = D_1 \cap \dots \cap D_{n-s}$, where $D_i \in |L|$ for $i = 1, \dots, n - s - 1$ are smooth and transversal and $D_{n-s} \in |mL|$ for some $m \geq 1$ is such that Y has ordinary singularities of multiplicity $s + 1$.*

Let Y be an irreducible variety of dimension s . A point $y \in Y$ is *ordinary of multiplicity m* ($m > 1$), if
 (i) the Zariski tangent space of Y at y has dimension $s + 1$, and
 (ii) the (affine) tangent cone to Y at y is a cone with vertex y over a smooth hypersurface of degree m in \mathbb{P}^s .

An ordinary point of Y is an isolated hypersurface singularity, hence, it is Gorenstein.

The proof of Theorem 0.1 is done in Section 1. In Section 2 we deduce an effective upper bound for gaps in the surface case. In Section 3, we focus on smooth surfaces in \mathbb{P}^3 , proving in particular that in this case there is no *absolute gap* for geometric genera of curves. That is, for all $d > 0$, all non-negative integers are geometric genera for some curves lying on some smooth surfaces of degree d in \mathbb{P}^3 .

Acknowledgments: The first and second authors have been supported by the Italian MIUR Project protocol 2010S47ARA_005 and by GNSAGA of INdAM. The third author was supported by the French-Italian cooperation project GRIFGA and by INdAM. The authors thank all Institutions which helped them in this collaboration, including their own Departments.

2010 Mathematics Subject Classification: 14N25, 14J70, 14C20, 14J29, 32Q45. *Key words:* projective hypersurface, geometric genus.

Notation and conventions. We work over the field of complex numbers and use standard notation and terminology. In particular, for X a reduced, irreducible, projective variety, we denote by ω_X its dualizing sheaf. We will sometimes abuse notation and use the same symbol to denote a divisor D on X and its class in $\text{Pic}(X)$. Thus K_X will denote a canonical divisor or the canonical sheaf ω_X . When $Y \subset X$ is a closed subscheme, $\mathcal{I}_{Y/X}$ will denote its ideal sheaf.

1. UPPER BOUND FOR GAPS

1.1. Preliminaries. In the sequel, X is an irreducible, complex projective variety of dimension $n \geq 2$. We assume usually that X is Gorenstein, so that ω_X is a line bundle. This holds, in particular, if X has only ordinary singularities. We set

$$p(X) := h^0(X, \omega_X) \text{ and } q(X) := h^1(X, \omega_X).$$

For smooth varieties, both $p(X)$ and $q(X)$ are birational invariants. Note that, if X is a smooth surface, then $q(X)$ is the *irregularity* of X .

The *geometric genus* of X is defined as

$$p_g(X) := p(X'),$$

where $X' \rightarrow X$ is any desingularization of X .

Lemma 1.1. *Let X be an irreducible, smooth projective variety of dimension n , and let Y be an irreducible, effective divisor on X . Assume that $h^i(X, \omega_X \otimes \mathcal{O}_X(Y)) = 0$ for all $i \geq 1$ ¹. Then:*

(i) *one has*

$$p(Y) = h^0(X, \omega_X \otimes \mathcal{O}_X(Y)) + q(X) - p_g(X),$$

which is the geometric genus if Y is smooth;

(ii) *suppose that $\text{Sing}(Y) = \{x_1, \dots, x_k\}$, where x_1, \dots, x_k are ordinary points of Y of multiplicity n . Then*

$$p_g(Y) \geq p(Y) - k,$$

and the equality holds if and only if x_1, \dots, x_k impose k independent conditions to the linear system $|\omega_X \otimes \mathcal{O}_X(Y)|$, i.e., if and only if the restriction map

$$H^0(X, \omega_X \otimes \mathcal{O}_X(Y)) \longrightarrow \bigoplus_{i=1}^k \mathcal{O}_{x_i} \quad (1)$$

is surjective.

Proof. Part (i) follows from the *adjunction sequence*

$$0 \longrightarrow \omega_X \longrightarrow \omega_X \otimes \mathcal{O}_X(Y) \longrightarrow \omega_X \otimes \mathcal{O}_X(Y) \otimes \mathcal{O}_Y \cong \omega_Y \longrightarrow 0.$$

As for part (ii), let $\pi : X' \rightarrow X$ be the blow-up of X at x_1, \dots, x_k with exceptional divisors E_1, \dots, E_k . Set $E = \sum_{i=1}^k E_i$. The union x of x_1, \dots, x_k is a 0-dimensional subscheme of X . The proper transform Y' of Y in X' is smooth and belongs to the linear system $|\pi^*(\mathcal{O}_X(Y)) \otimes \mathcal{O}_{X'}(-nE)|$, whereas $\omega_{X'} = \pi^*(\omega_X) \otimes \mathcal{O}_{X'}((n-1)E)$. Hence, by (i), one has

$$\begin{aligned} p_g(Y) &= p_g(Y') = h^0(X', \omega_{X'} \otimes \mathcal{O}_{X'}(Y')) + q(X') - p_g(X') \\ &= h^0(X', \pi^*(\omega_X \otimes \mathcal{O}_X(Y)) \otimes \mathcal{O}_{X'}(-E)) + q(X) - p_g(X) \\ &= h^0(X, \omega_X \otimes \mathcal{O}_X(Y) \otimes \mathcal{I}_{x/X}) + q(X) - p_g(X). \end{aligned}$$

Now the assertion follows. □

Lemma 1.2. *Let $X \subset \mathbb{P}^r$ be a non-degenerate, irreducible projective variety of dimension n . Let $x_1, \dots, x_k \in X$ be general points. If $k \leq r - n = \text{codim}_{\mathbb{P}^r}(X)$, then the scheme theoretical intersection of the linear space $\langle x_1, \dots, x_k \rangle$ with X is the reduced 0-dimensional scheme consisting of x_1, \dots, x_k .*

¹By the Kawamata–Viehweg vanishing theorem this holds provided Y is nef and big (in particular, for Y ample).

Proof. The assertion is trivial for $k = 1$, so we assume $k \geq 2$. For $n = 1$ and $k = 2$, this is the classical *trisecant lemma*, to the effect that a general chord of a non-degenerate curve in \mathbb{P}^r , where $r \geq 3$, is not a trisecant (see, e.g., [1, Example 1.8] for a simple proof). If $n = 1$ and $k > 2$, one proceeds by applying induction on k to the projection of X to \mathbb{P}^{r-1} from one of the points x_1, \dots, x_k .

If $n > 1$, one proceeds by applying induction on n to the section of X with a general hyperplane containing $\langle x_1, \dots, x_k \rangle$. \square

1.2. The theorem.

Theorem 1.3. *Let X be an irreducible, smooth, projective variety of dimension $n > 1$, and let L be a very ample line bundle on X . Then there is an integer $p_{X,L}$ (depending on X and L) such that for all $p \geq p_{X,L}$ one can find an irreducible hypersurface $Y \in \bigcup_{m \geq 1} |mL|$ with at most ordinary points of multiplicity n as singularities and with $p_g(Y) = p$.*

Proof. Set $d := L^n$. For a positive integer m we denote by p_m the geometric genus of smooth elements in $|mL|$ (which is of course a non-gap). We show that for m sufficiently large, any integer p in the interval $[p_{m-1} + 1, p_m - 1]$ is the geometric genus of a hypersurface in $|mL|$ with $p_m - p$ ordinary points of multiplicity n as singularities, which can be taken generically on X .

Since L is very ample, by Lemma 1.1–(i) and by the asymptotic Riemann–Roch Theorem [6, Vol. I, p. 21], we have

$$\begin{aligned} p_m &= \chi(\omega_X \otimes \mathcal{O}_X(mL)) + q(X) - p_g(X) \\ &= h^0(\omega_X \otimes \mathcal{O}_X(mL)) + q(X) - p_g(X) \\ &= \frac{m^n}{n!}d + O(m^{n-1}). \end{aligned} \tag{2}$$

Hence

$$\delta_m := p_m - p_{m-1} - 1 = \frac{m^{n-1}}{(n-1)!}d + O(m^{n-2}). \tag{3}$$

Theorem 1.3 follows from the:

Claim 1. *There is an integer $m_{X,L}$ (depending on X and L) such that for all $m \geq m_{X,L}$, for all positive integers $k \leq \delta_m$, and for general points x_1, \dots, x_k in X , one can find an irreducible element $Y \in |mL|$ with ordinary points of multiplicity n at x_1, \dots, x_k and no other singularity.*

Indeed, suppose that Claim 1 holds. Then the map (1) is surjective by the generality of x_1, \dots, x_k . Thus Lemma 1.1–(ii) implies Theorem 1.3 with

$$p_{X,L} := p_{m_{X,L}-1}.$$

In turn, Claim 1 is a consequence of the following

Claim 2. *There is an integer $m_{X,L} \geq n$ such that for all $m \geq m_{X,L}$, one has*

$$\delta_m \leq \dim(|\nu L|) - n, \quad \text{where } m = n\nu + \mu \quad \text{with } \mu \in \{0, \dots, n-1\}. \tag{4}$$

Indeed, assuming that Claim 2 holds, let x be the reduced 0-dimensional scheme formed by the points x_1, \dots, x_k , and let $\Lambda := \nu L \otimes \mathcal{I}_{x/X}$. By Lemma 1.2, (4) ensures that x is the base locus scheme of the linear system $|\Lambda|$. Therefore, by Bertini's theorem the general $Y \in |\Lambda^{\otimes n} \otimes \mathcal{O}_X(\mu)| \subset |mL|$ is irreducible having x_1, \dots, x_k as ordinary points of multiplicity n and no other singularity. Thus, Claim 2 implies Claim 1.

Finally, we prove Claim 2.

Proof of Claim 2. By the asymptotic Riemann–Roch Theorem (cf. (2)), one has

$$\dim(|\nu L|) = \frac{\nu^n}{n!}d + O(\nu^{n-1}).$$

Hence, by (3), Claim 2 holds if, for $m \gg 0$, one has

$$n m^{n-1} < \nu^n. \tag{5}$$

Since $\frac{m}{n} < \nu + 1$, (5) is true for $m \gg 0$. □

This ends the proof of Theorem 1.3. □

Remark 1.4. As follows from the proof, the upper bound $p_{X,L}$ depends only on the Hilbert polynomial of $\bigoplus_{m \geq 1} H^0(X, \omega_X \otimes \mathcal{O}_X(mL))$ and of $\bigoplus_{m \geq 1} H^0(X, \mathcal{O}_X(mL))$. The former coincides with the Hilbert function by Kodaira's Theorem. Assuming that $h^i(X, \mathcal{O}_X(mL)) = 0$ for all positive integers m and i , it is possible to replace the asymptotic Riemann–Roch theorem with the true Riemann–Roch, which is then purely numerical. This gives in principle an effective bound on the integers $m_{X,L}$ and $p_{X,L}$ in Theorem 1.3 (cf. Section 2 for a particular case).

Proof of Theorem 0.1. With X , L , n , and s as in Theorem 0.1, it suffices to apply Theorem 1.3 to $X' = D_1 \cap \dots \cap D_{n-s-1}$ instead of X and $L|_{X'}$ instead of L , where $D_1, \dots, D_{n-s-1} \in |L|$ are general. □

2. GENERA OF CURVES ON SMOOTH SURFACES

In this section we compute an effective upper bound for gaps of geometric genera of curves on surfaces.

Theorem 2.1. *Let S be a smooth, irreducible, projective surface, and L a very ample line bundle on S . Set*

$$p := p_g(S), \quad q := q(S), \quad d := L^2, \quad \text{and} \quad e := K_S \cdot L.$$

For $\varepsilon \in \{0, 1\}$, set

$$\Delta(\varepsilon) := 4(3 + 2\varepsilon)d^2 + 12de + e^2 - 8d(p - q), \tag{6}$$

$$n_1 = n_1(\varepsilon) := \begin{cases} 2 & \text{if } \Delta(\varepsilon) < 0, \\ \left\lceil 4 + \varepsilon + \frac{e}{d} + \sqrt{\frac{\Delta(\varepsilon)}{d^2}} \right\rceil & \text{if } \Delta(\varepsilon) \geq 0, \end{cases} \tag{7}$$

$$n_2 = n_2(\varepsilon) := \left\lceil \frac{6(p - q) + d(1 + \varepsilon) + e(2\varepsilon - 1) - 12}{e + 2d(1 + \varepsilon)} \right\rceil, \tag{8}$$

$$n_3 := \min \left\{ n \in \mathbb{N} \mid \left\lfloor \frac{n}{2} \right\rfloor^2 d > nd - \frac{d - e}{2} - 1 \right\}, \tag{9}$$

$$n_4 := \min \{ n \in \mathbb{N} \mid h^1(S, \mathcal{O}_S(nL)) = h^2(S, \mathcal{O}_S(nL)) = 0 \}, \tag{10}$$

and

$$n_0 = n_0(\varepsilon) := \max\{n_1(\varepsilon), n_2(\varepsilon), n_3, n_4\}. \tag{11}$$

Set finally

$$\varphi(d, e, n_0) = \frac{1}{2} [(n_0 - 1)((n_0 - 1)d + e)] + 1. \tag{12}$$

Then for any $g \geq \varphi(d, e, n_0)$ the surface S carries a reduced, irreducible curve C of geometric genus g with only nodes as singularities.

The proof of Theorem 2.1 is basically the same as the one of Theorem 1.3 in the case of surfaces, with a slight improvement, based upon the following:

Theorem 2.2. ([1, Thm. 1.4], [5, Thm. 1.3]) *Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non-degenerate variety of dimension m . Assume X is not k -weakly defective for a given $k \geq 0$ such that*

$$r \geq (m + 1)(k + 1). \tag{13}$$

Then, given general points p_0, \dots, p_k on X , the general hyperplane H containing T_{X, p_0, \dots, p_k}^2 is tangent to X only at p_0, \dots, p_k . Such a hyperplane H cuts out on X a divisor with ordinary double points at p_0, \dots, p_k and no further singularities.

² T_{X, p_0, \dots, p_k} stands for the linear span of the union of the embedded tangent spaces T_{X, p_i} , $i = 0, \dots, k$.

Recall (see [3, p.152]) that a variety X as in Theorem 2.2 is said to be *k-weakly defective* if, given $p_0, \dots, p_k \in X$ general points and a general hyperplane H containing T_{X,p_0,\dots,p_k} (i.e., *tangent* to X at p_0, \dots, p_k), then H cuts out on X a divisor H_X such that there is a positive dimensional subvariety $\Sigma \subseteq \text{Sing}(H_X)$ containing p_0, \dots, p_k (Σ is then called the *contact variety of H*).

2.1. Proof of Theorem 2.1. The arithmetic genus of curves in $|nL|$ is

$$p(d, e, n) := \frac{1}{2}n(nd + e) + 1. \quad (14)$$

For $n \geq n_0$ set

$$l(d, e, n) := \dim(|nL|) = \frac{1}{2}n(nd - e) + p - q, \quad (15)$$

where the latter equality follows by the Riemann–Roch Theorem and (10), since we assume $n \geq n_0 \geq n_4$. Consider the embedding

$$\varphi_{|nL|}: S \hookrightarrow \mathbb{P}^{l(d,e,n)}.$$

Since $\varphi_{|nL|}$ is an isomorphism of S to its image S_n , we may identify S with S_n .

Set

$$\delta(d, e, n) := p(d, e, n) - p(d, e, n-1) - 1 = nd - \frac{1}{2}(d - e) - 1. \quad (16)$$

As in the proof of Theorem 1.3, we show that for any $n \geq n_0$ and any positive integer $k \leq \delta(d, e, n) - 1$, one can find an irreducible curve $C \in |nL|$ with exactly $k+1$ nodes at general points of S as its only singularities. Then, for any $n \geq n_0$, all the integers in the interval $J_n = [p(d, e, n-1), p(d, e, n)]$ are non-gaps. Since the intervals J_n and J_{n+1} overlap, this proves Theorem 2.1, because

$$\varphi(d, e, n_0) := \min(J_{n_0}) = p(d, e, n_0 - 1) \quad (17)$$

is exactly (12).

The proof follows by Proposition 2.3 and Lemma 2.4 below (which are of independent interest).

Proposition 2.3. *Let S be a smooth, irreducible, projective surface, and L a very ample line bundle on S . Assume that $n \geq \max\{n_3, 2\}$ and that (with the above notation) the following inequalities hold*

$$l(d, e, n) \geq 3(\delta(d, e, n) - 1) \quad (18)$$

and

$$l(d, e, \lfloor n/2 \rfloor) \geq \delta(d, e, n) + 1. \quad (19)$$

Then for any $k \in \{0, \dots, \delta(d, e, n) - 1\}$,

- (a) the smooth surface $S_n \subset \mathbb{P}^{l(d,e,n)}$ is not *k-weakly defective*, and
- (b) there exists a reduced, irreducible curve $C \in |nL|$ in S with nodes at $k+1$ general points of S and no other singularity.

Proof. Let x_0, \dots, x_k be general points of S . Inequality (19) guarantees that, for any $k \in \{0, \dots, \delta(d, e, n) - 1\}$, one has

$$\dim |\mathcal{O}_S(\lfloor n/2 \rfloor L) \otimes \mathcal{I}_{\{x_0, \dots, x_k\}/S}| = l(d, e, \lfloor n/2 \rfloor) - k - 1 \geq l(d, e, \lfloor n/2 \rfloor) - \delta(d, e, n) \geq 1.$$

The general curve in $|\mathcal{O}_S(\lfloor n/2 \rfloor L) \otimes \mathcal{I}_{\{x_0, \dots, x_k\}/S}|$ is reduced and irreducible. Letting C_1 and C_2 be two different such general curves, and C_0 a general member of L , we obtain a divisor

$$C = \varepsilon C_0 + C_1 + C_2 \in |\mathcal{O}_S(nL) \otimes \mathcal{I}_{T_{S,x_0,\dots,x_k}/\mathbb{P}^{l(d,e,n)}}|,$$

where $\varepsilon \in \{0, 1\}$, $\varepsilon \equiv n \pmod{2}$. Since C is reduced, with nodes at x_0, \dots, x_k , this shows that (a) holds.

Now (b) follows. Indeed, since $k+1 \leq \delta(d, e, n)$, (18) yields (13) with $m = 2$ and $r = l(d, e, n)$. Hence Theorem 2.2 applies, and so, the general curve in $|\mathcal{O}_S(nL) \otimes \mathcal{I}_{T_{S,x_0,\dots,x_k}/\mathbb{P}^{l(d,e,n)}}|$ has nodes at x_0, \dots, x_k and is elsewhere smooth. This curve is irreducible by Bertini's theorem. Indeed, if n is odd, then $|\mathcal{O}_S(nL) \otimes$

$\mathcal{I}_{T_{S,x_0,\dots,x_k}/\mathbb{P}^l(d,e,n)}|$ has no fixed component and is not composed with a pencil. Assume that n is even. By (9),

$$C_1 \cdot C_2 = \frac{n^2}{4}d > \delta(d,e,n) \geq k+1,$$

which motivates (9). So, the general curve in $|\mathcal{O}_S(nL) \otimes \mathcal{I}_{T_{S,x_0,\dots,x_k}/\mathbb{P}^l(d,e,n)}|$, being singular only at x_0, \dots, x_k , cannot be of the form $C_1 + C_2$, hence it must be irreducible. \square

Lemma 2.4. *Let $\varepsilon \in \{0, 1\}$ be such that $\varepsilon \equiv n \pmod{2}$, and let $\Delta(\varepsilon)$ be as in (6). Then*

(a) (19) holds for any $n \geq n_1$, with n_1 as in (7);

(b) if (19) holds, then also (18) holds, provided that $n \geq n_2$, with n_2 as in (8).

Proof. (a) Write $n = 2t + \varepsilon$, with $t \geq 1$ since $n \geq n_1 \geq 2$. From (15) and (16), (19) reads

$$t^2d - t(4d + e) + 2(p - q) - e + (1 - 2\varepsilon)d \geq 0. \quad (20)$$

and the discriminant of the left hand side is $\Delta(\varepsilon)$ as in (6).

When $\Delta(\varepsilon) \geq 0$, (20) holds for $t \geq \frac{4d+e+\sqrt{\Delta(\varepsilon)}}{2d}$, and so (19) holds for

$$n \geq 4 + \varepsilon + \frac{e}{d} + \sqrt{\frac{\Delta(\varepsilon)}{d^2}}.$$

If $\Delta(\varepsilon) < 0$, then (19) holds for any $n \geq 2$. This motivates the definition of n_1 in (7) and proves (a).

(b) As above, (18) reads

$$n^2d - n(6d + e) + 2(p - q) + 3(d - e) + 12 \geq 0. \quad (21)$$

Moreover, (20) reads

$$n^2d - 8nd - 2ne + 8(p - q) + 4(d - e) + \varepsilon(\varepsilon d - 2nd + 2e) \geq 0. \quad (22)$$

The difference between the left hand side in (21) and that of (22) is

$$2nd + ne - 6(p - q) - (d - e) - \varepsilon(\varepsilon d - 2nd + 2e) + 12,$$

which is non-negative as soon as

$$n \geq \frac{6(p - q) + d(1 + \varepsilon) + e(2\varepsilon - 1) - 12}{e + 2d(1 + \varepsilon)}.$$

Assuming (a), this motivates the definition of n_2 in (8) and proves (b). \square

Proof of Theorem 2.1. The integer n_0 in (11) satisfies both (a) and (b) in Lemma 2.4. Hence (18) and (19) hold, and we can conclude by Proposition 2.3. \square

3. GENERA OF CURVES ON SMOOTH SURFACES IN \mathbb{P}^3

Here we focus on the case S is a smooth surface of degree $d \geq 4$ in \mathbb{P}^3 . In [4] we considered the case of a very general $S \in |\mathcal{O}_{\mathbb{P}^3}(d)|$; here we drop this assumption, and simply assume S smooth and $d \geq 4$ (the case $d < 4$ being trivial for our considerations, because then S carries curves of any genus). As a direct consequence of Theorem 2.1, we have:

Corollary 3.1. *For any integer $d \geq 4$ there exists an integer c_d such that, for any smooth surface S in \mathbb{P}^3 of degree d and any integer $g \geq c_d$, S carries a reduced, irreducible nodal curve of geometric genus g , whose nodes can be prescribed generically on S .*

One can give an effective upper bound for c_d . We keep here the notation of Section 2. Letting $L = \mathcal{O}_S(1)$ we obtain

$$e = d(d - 4), \quad q = q(S) = 0, \quad \text{and} \quad p = p_g(S) = \frac{1}{6}(d - 1)(d - 2)(d - 3).$$

By Theorem 2.1 one has

$$c_d \leq \varphi(d, d(d - 4), n_0), \quad (23)$$

cf. (12). Thus, we are left to compute n_0 as in (11). Since, by Serre duality, $n_4 = d - 3$, this amounts to compute n_1, n_2 , and n_3 as in (7), (8), and (9).

From (6) we get

$$\Delta(\varepsilon) = d \left(-\frac{1}{3}d^3 + 12d^2 - \frac{1}{3}(104 - 24\varepsilon)d + 8 \right).$$

The polynomial $\Delta(\varepsilon)/d$ has three positive roots

$$d_1, d_2, d_3 \sim \begin{cases} 0, 25, 2, 89, 32, 86 & \text{if } \varepsilon = 0, \\ 0, 36, 2, 33, 64 & \text{if } \varepsilon = 1. \end{cases}$$

Thus, $\Delta(0) \geq 0$ for $4 \leq d \leq 32$ and $\Delta(0) \leq 0$ for $d \geq 33$, while $\Delta(1) \geq 0$ for $4 \leq d \leq 33$ and $\Delta(1) \leq 0$ for $d \geq 34$.

Now (7), (8), and (9) give, respectively,

$$\begin{aligned} n_1(0) &= \begin{cases} 2 & \text{if } d \geq 33, \\ \left\lceil d + \sqrt{\frac{\Delta(0)}{d^2}} \right\rceil & \text{if } 4 \leq d \leq 32, \end{cases} & n_1(1) &= \begin{cases} 2 & \text{if } d \geq 34, \\ \left\lceil d + 1 + \sqrt{\frac{\Delta(1)}{d^2}} \right\rceil & \text{if } 4 \leq d \leq 33, \end{cases} \\ n_2(0) &= \left\lceil d - 5 + \frac{6(d-3)}{d(d-2)} \right\rceil, & n_2(1) &= \left\lceil d - 5 + \frac{9(d-2)}{d^2} \right\rceil, \end{aligned}$$

and

$$n_3(0) = 3 + \left\lfloor \sqrt{2d - 6 - (4/d)} \right\rfloor, \quad n_3(1) = 4 + \left\lfloor \sqrt{2d - 2 - (4/d)} \right\rfloor.$$

In particular, for $d \gg 0$, one has

$$n_1 = 2, \quad n_2 \sim d - 5, \quad n_3 \sim \sqrt{d}, \quad \text{hence} \quad n_0 = n_4 = d - 4.$$

So, by (12) and (23),

$$c_d \leq \varphi(d, d(d-4), d-4) = \frac{d(d-5)(2d-9)}{2} \sim d^3.$$

Remark 3.2. Let $\text{Gaps}(d)$ be the set of gaps for geometric genera of irreducible curves on $S \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ very general. By [4, Theorem 2.4], one has

$$\text{Gaps}(4) = \emptyset, \quad \text{Gaps}(5) = \{0, 1, 2\}, \quad \text{and} \quad \text{Gaps}(d) \subset \left[0, \frac{d(d-1)(5d-19)}{6} - 1 \right] \quad \text{for } d \geq 6.$$

This is compatible with the results of the present section. A more refined analysis based on [4, Remark 2.5], shows that the maximum G_d of $\text{Gaps}(d)$ goes like $G_d = O(d^{\frac{8}{3}})$. It is an open problem to see if this is sharp.

3.1. Absence of absolute gaps for curves on smooth surfaces in \mathbb{P}^3 . We say that an integer g is a *d-absolute gap* if there is no irreducible curve with geometric genus g on any smooth surface of degree d . We show here that there is no absolute gap at all.

Theorem 3.3. *For any positive integer d and for any non-negative integer g , there is a smooth surface $S \subset \mathbb{P}^3$ of degree d and an irreducible, nodal curve C on S with geometric genus g .*

Proof. We may assume $d \geq 5$, otherwise the result is well known (cf. e.g. [4, Prop. 1.2 and Cor. 2.2]).

We set

$$\ell_{d,n} := l(d, d(d-4), n) = \begin{cases} \frac{n(n^2+6n+11)}{6} & \text{if } n < d \\ \frac{d(3n^2-3n(d-4)+(d^2-6d+11))}{6} - 1 & \text{if } n \geq d, \end{cases} \quad (24)$$

and

$$p_{d,n} := p(d, d(d-4), n) = \frac{dn(d+n-4)}{2} + 1. \quad (25)$$

By [4, Thm. 2.4 and Rem. 2.5], for $S \subset \mathbb{P}^3$ very general one has

$$\text{Gaps}(d) \subset [0, p_{d,n-1} - \ell_{d,n-1} - 1] = \left[0, \frac{d(d-1)(5d-19)}{6} - 1\right] \text{ if } d > n \geq \sqrt[3]{12d^2}.$$

Plugging $n = d - 1$ in this formula, we obtain the desired result for all

$$g \geq p_{d,d-2} - \ell_{d,d-2}.$$

Take now $n \leq d - 2$. By [2, Theorem 3.1], for a general surface $\Sigma \subset \mathbb{P}^3$ of degree n with $4 \leq n \leq d - 2$ and for any $g \in [p_{n,d} - \ell_{n,d}, p_{n,d}]$ there is a reduced, irreducible component \mathcal{V} of the Severi variety of complete intersections of Σ with surfaces of degree d having $\delta = p_{n,d} - g$ nodes as the only singularities.

Notice that the union of integers in the non-gap intervals

$$J_{d-1}(n) = [p_{n,d-1} - \ell_{n,d-1}, p_{n,d-1}] \text{ and } J_d(n) = [p_{n,d} - \ell_{n,d}, p_{n,d}]$$

is an integer interval for $n \leq d - 2$. To see this, it suffices to observe that

$$p_{n,d} - \ell_{n,d} \leq p_{n,d-1} + 1 < p_{n,d}.$$

The inequality on the right is trivial. To show the other inequality

$$\ell_{n,d} \geq p_{n,d} - p_{n,d-1} - 1,$$

using (24), (25), and the fact that $d \geq n + 2 > n$, we can rewrite it as

$$3d(d - n + 2) + (n^2 - 9n + 26) \geq 0,$$

which holds if $d \geq n + 2$.

Any curve $C \in \mathcal{V}$ is cut out on Σ by a surface S of degree d . We claim that S can be taken to be smooth. Since the linear system $|\mathcal{I}_{C/\mathbb{P}^3}(d)|$ is base point free outside C , by Bertini's theorem S can be chosen to be smooth off C . Suppose S is singular at a point $p \in C$. Since C has δ nodes and no other singularity, and it is the complete intersection of S and Σ , then p is a node of C . But the general surface in $|\mathcal{I}_{C/\mathbb{P}^3}(d)|$ is non-singular at the nodes of C , because $|\mathcal{I}_{C/\mathbb{P}^3}(d)|$ contains all surfaces of the form $\Sigma + \Phi$, where Φ is a general surface of degree $d - n$ (so it does not contain the nodes of C), and Σ is smooth (thus $\Sigma + \Phi$ is smooth at the nodes of C).

In this way we find nodal curves of any geometric genus $g \geq p_{4,d} - \ell_{4,d} = 0$ on smooth surfaces of degree d , proving the assertion. \square

Let us conclude by the following conjecture.

Conjecture. *For any smooth, rational variety X of dimension $n + 1$, any very ample line bundle L on X , any $s \in \{1, \dots, n - 1\}$, and any integer $g \geq 0$ there is a smooth hypersurface $D \in |\mathcal{O}_X(L)|$ carrying an s -dimensional subvariety $S \subset D$ of geometric genus g .*

In particular, for any $n \geq 3$, $d \geq 1$, $s \in \{1, \dots, n - 1\}$, and $g \geq 0$ there is a smooth hypersurface $D \in |\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ and a subvariety $S \subset D$ as before.

One can ask whether the same holds, more generally, for any smooth Fano variety X .

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